

# On positional strategies over finite arenas

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## **Perfect information games**

Finite duration games (like chess) can be presented as games on graphs.

Complexity of solving such games relies on the structure of the graph ( $\rightarrow$  alternating reachability).

**Infinite** duration games are usually modelled as games on **colored** graphs.

Complexity relies on the structure of both: the graph and the **winning condition**.

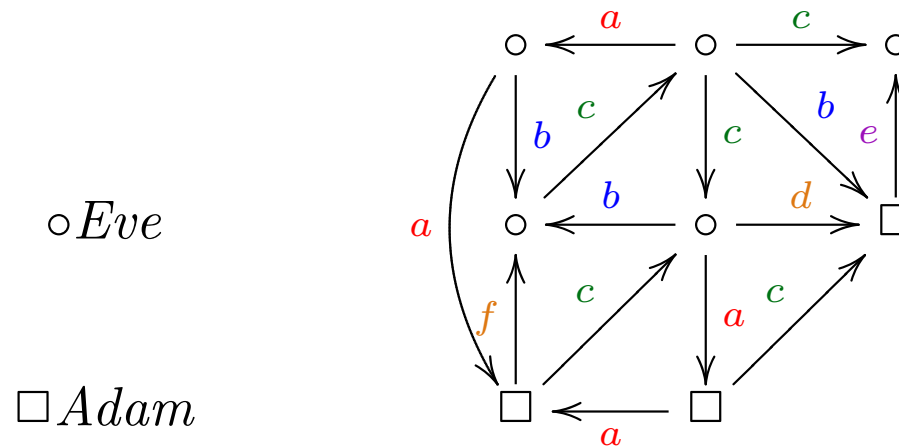
## Games on (edge colored) graphs

$$G = \langle Pos_{\exists}, Pos_{\forall}, Move, C, rank, W_{\exists}, W_{\forall} \rangle,$$

where  $Pos = Pos_{\exists} \dot{\cup} Pos_{\forall}$ ,  $Move \subseteq Pos \times Pos$ ,

$rank : Move \rightarrow C$ ,

$W_{\exists}, W_{\forall} \subseteq C^{\omega}$ ,  $W_{\forall} \cap W_{\exists} = \emptyset$ .



Player who cannot move, loses — the opponent wins.

An infinite play  $p_0, p_1, \dots$  is won by  $Q$  iff  $rank(p_0, p_1), rank(p_1, p_2) \dots \in W_Q$ .

Otherwise there is a draw.

## Strategies

A strategy (for Eve, say) is a partial mapping  $Move^* \rightarrow Move$  defined for paths ending in a position of Eve.

It is **winning** if any play  $\pi$  consistent with the strategy is won by Eve.

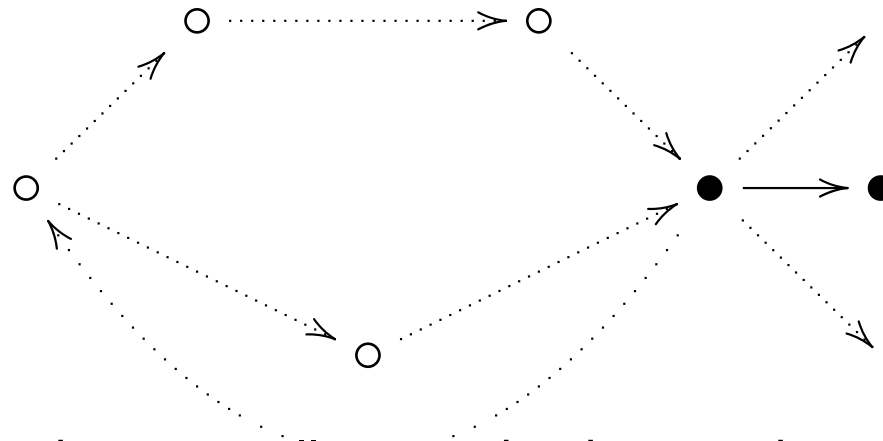
A game is **determined** if, for any position, one of the players has a winning strategy, or both players have strategies to achieve (at least) a draw.

**Reachability game:** No colors. Infinite play is always a draw.

**Zermelo's theorem:** Reachability games are determined.

## Positional strategies

A **positional** strategy depends only on the actual position.



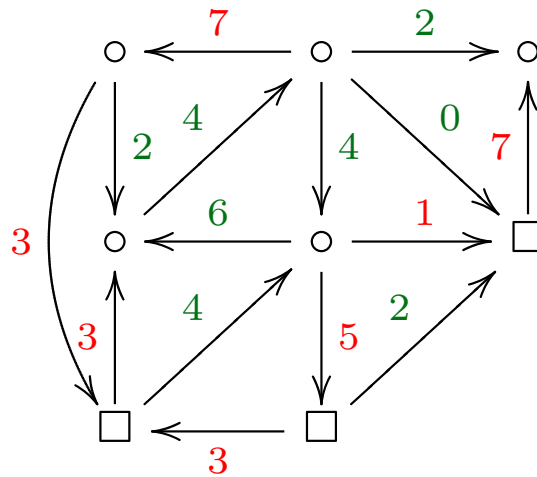
Positional determinacy — all strategies in question are positional.

Reachability games are **positionally determined** (on all graphs).

What else?

## Parity games

$$C = \{0, 1, \dots, n\}.$$



Eve wants  $\infty$  **even**, Adam wants  $\infty$  **odd**, **maximal** wins.

$$W_{\exists} = \{u \in C^{\omega} : \limsup_{n \rightarrow \infty} u_n \text{ is } \mathbf{even} \}$$

$$W_{\forall} = \{u \in C^{\omega} : \limsup_{n \rightarrow \infty} u_n \text{ is } \mathbf{odd} \}.$$

Parity games are positionally determined on all graphs (**Emerson & Jutla 1991**, **Mostowski 1991**).

Essentially, it is the **only** condition with this property.

Suppose that  $W \subseteq C^\omega$  is uniform ( $W = CW$ ), and any game

$$\langle Pos_{\exists}, Pos_{\forall}, Move, C, rank, W, \overline{W} \rangle$$

is positionally determined. Then  $W$  is a parity condition **up to renaming the letters** (not necessarily 1:1). That is, there is  $n$  and  $h : C \rightarrow \{0, 1, \dots, n\}$ , such that

$$u \in W \quad \text{iff} \quad \limsup_{i \rightarrow \infty} h(u_i) \text{ is even}$$

(**Colcombet & N. 2006**).

## Positional determinacy over finite graphs

There are more conditions that guarantee positional determinacy. For example

$$W = \{x \in \{0, 1\}^\omega : \lim_{n \rightarrow \infty} \frac{x_1 + \dots + x_n}{n} = 0\}$$

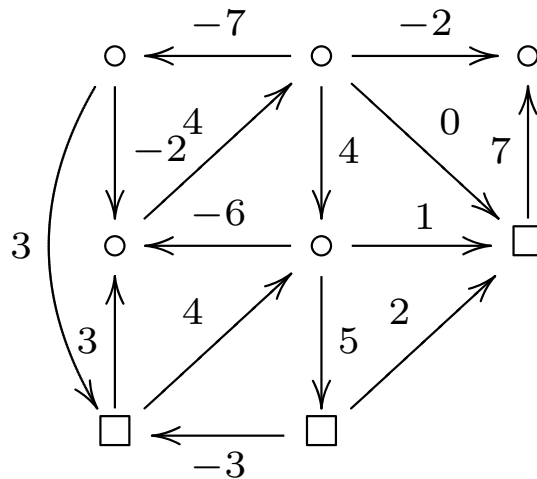
Clearly,  $W$  cannot be renamed to parity condition.

(Besides, it is  $\mathbf{\Pi}_3^0$ -complete, whereas the parity conditions are in  $\mathbf{\Delta}_3^0$ .)

Positional determinacy of games on finite graphs with  $W_\exists = W$ ,  $W_\forall = \overline{W}$  follows from a more general property.



## Mean-payoff optimization games (over finite arenas)



Adam pays to Eve the amount  $q$ , while passing through an edge  $\xrightarrow{q}$ .

Each player wants to maximize her/his income asymptotically on average.

For each position  $p$ , there is a **compromise value**  $val(p)$ , which Eve and Adam can reach using **positional strategies** (Ehrenfeucht & Mycielski 1979).

More specifically, let, for a play  $\pi = (p_0, p_1, \dots)$  and  $n \geq 1$ ,

$$val_n(\pi) = \frac{rank(p_0, p_1) + rank(p_1, p_2) \dots + rank(p_{n-1}, p_n)}{n}.$$

Let  $play(s_E, s_A, p)$  be a unique play determined by strategies  $s_E$  and  $s_A$ , and position  $p$ .

**Ehrenfeucht & Mycielski 1979** show that, for any  $p$ , there are positional strategies  $\overline{s_E}, \overline{s_A}$ , such that

$$val(p) =_{def} \lim_{n \rightarrow \infty} val_n(play(\overline{s_E}, \overline{s_A}, p)),$$

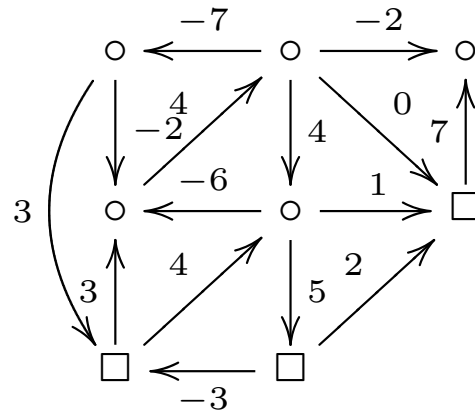
satisfies

$$\begin{aligned} val(p) &= \inf_{s_A} \sup_{s_E} \limsup val_n(play(s_E.s_A, p)) \\ &= \sup_{s_E} \inf_{s_A} \liminf val_n(play(s_E.s_A), p) \end{aligned}$$

where  $s_E, s_A$  range over all strategies.

## Mean-payoff winning conditions

$C \subseteq \mathbb{Z}$  (finite).



For a fixed threshold  $d$ ,

$$W_{\exists} = \left\{ x : \liminf \frac{x_1 + \dots + x_n}{n} \geq d \right\}$$

$$W_{\forall} = \left\{ x : \limsup \frac{x_1 + \dots + x_n}{n} < d \right\}.$$

## Can we characterize positional determinacy on finite arenas by a class of winning conditions?

(Such a class should somehow subsume parity games.)

**Gimbert 2006** gave elegant structural conditions that characterize positional determinacy (not necessarily uniform) on all finite graphs.

**Note.** For finite arenas, winning conditions may admit various presentations.

## Equivalence of winning conditions

For example, the aforementioned condition

$$W = \{x \in \{0, 1\}^\omega : \lim_{n \rightarrow \infty} \frac{x_1 + \dots + x_n}{n} = 0\}$$

is over finite arenas equivalent to

$$W' = \{x \in \{0, 1\}^\omega : \liminf_{n \rightarrow \infty} x_n = 0\}$$

More generally,

**Periodicity lemma.** Two winning conditions  $(W_\exists, W_\forall)$  and  $(W'_\exists, W'_\forall)$  are equivalent over finite arenas iff they contain the same ultimately periodic words.

## **Proof of the lemma.**

**(only if)**

If an ultimately periodic word  $u$  separates the two conditions, we can take a game that essentially consists of this word.

**(if)**

Let  $s_E$  be a positional strategy for Eve winning from position  $p$  with the condition  $(W_{\exists}, W_{\forall})$ .

Suppose Adam has a positional strategy  $s'_A$  from  $p$  to achieve at least a draw with the condition  $(W'_{\exists}, W'_{\forall})$ .

Then the labeling of  $play(s_E, s'_A, p)$  separates the two conditions.

## Some consequences of periodicity lemma.

### Parity vs. boundedness

If  $|w|_a$  denotes the number of occurrences of  $a$  in  $w$ , let

$$W'_{\exists} = \{u : (\exists M \forall a \text{ odd} \in C) |w_a| \leq M, \text{ where } w \text{ ranges over all finite factors of } u \text{ s.t. the maximal color of } w \text{ is } a\}$$

$$W'_{\forall} = \{u : (\exists M \forall b \text{ even} \in C) |w_b| \leq M, \text{ where } w \text{ ranges as above up to } a \rightleftharpoons b\}$$

Then  $C^\omega - (W'_{\exists} \cup W'_{\forall}) \neq \emptyset$ , but any game on finite arena with the winning condition  $(W'_{\exists}, W'_{\forall})$  is equivalent to parity game, cf. [Colcombet & Loeding 2009](#).

## What can we gain by that?

If a winning condition  $(W'_\exists, W'_\forall)$  is equivalent to  $(W_\exists, W_\forall)$ , for some  $W'_\exists \subseteq W_\exists$  and  $W'_\forall \subseteq W_\forall$  then it is the same for any **separating pair**  $(\mathbf{W}''_\exists, \mathbf{W}''_\forall)$ , i.e.,

$$W'_\exists \subseteq \mathbf{W}''_\exists \subseteq W_\exists$$

$$W'_\forall \subseteq \mathbf{W}''_\forall \subseteq W_\forall.$$

This may have impact on complexity if  $\mathbf{W}''_\exists$  and  $\mathbf{W}''_\forall$  are simpler than the original condition. Cf. [Calude et al. 2017](#), and [Bojańczyk & Czerwiński 2018](#).

Specifically, for games with  $< \mathbf{M}$  positions, there is a “simple” separator of

$$W_\exists^{(\mathbf{M})} = \{u : (\forall a \text{ odd}) |w_a| \leq \mathbf{M}\}$$

$$W_\forall^{(\mathbf{M})} = \{u : (\forall b \text{ even}) |w_b| \leq \mathbf{M}\}$$

where  $w$  ranges as above.



## Example: intrinsically non-regular mean-payoff condition

$$W_{\exists} = \{x \in \{-1, 0, 1\}^{\omega} : \liminf_{n \rightarrow \infty} \frac{x_1 + \dots + x_n}{n} > 0\}$$

$$W_{\forall} = \{x \in \{-1, 0, 1\}^{\omega} : \limsup_{n \rightarrow \infty} \frac{x_1 + \dots + x_n}{n} \leq 0\}.$$

There is no  $\omega$ -regular language  $L$ , such that  $W_{\exists}$  and  $L$  contain the same ultimately periodic words.

**Note.** For an ultimately periodic word  $x \in \mathbb{Z}^{\omega}$ ,

$$\liminf_{n \rightarrow \infty} \frac{x_1 + \dots + x_n}{n} > 0 \text{ iff } \lim_{n \rightarrow \infty} x_1 + \dots + x_n = +\infty.$$

## In the search of a characterization

Let, for  $x \in \mathbb{Z}^\omega$ ,

$$\chi(x) = \begin{cases} 1 & \text{if } \lim_{n \rightarrow \infty} x_1 + \dots + x_n = +\infty \\ -1 & \text{if } \lim_{n \rightarrow \infty} x_1 + \dots + x_n = -\infty \\ 0 & \text{otherwise} \end{cases}$$

For  $x = (x^{(1)}, \dots, x^{(k)}) \in (\mathbb{Z}^k)^\omega$ , let

$$\vec{\chi}(x) = \left( \chi(x^{(1)}), \dots, \chi(x^{(k)}) \right)$$

The **lexicographic energy condition**:

$$\vec{\chi}(x) >_{lex} \vec{0}.$$

## Properties

Let  $W_{\exists}^C$  be the set of words in  $(\mathbb{Z}^k)^\omega$  satisfying the LE condition over the alphabet  $C \subseteq \mathbb{Z}^k$ .

Let  $W_{\forall}^C = \overline{W_{\exists}^C}$ .

The LE condition guarantees **positional determinacy** over finite arenas.

It subsumes mean-payoff ( $k = 1$ ), as well as parity:

rank

$$0 \rightarrow (0, 0, 0, 0, 1)$$

$$1 \rightarrow (0, 0, 0, -1, 0)$$

$$2 \rightarrow (0, 0, 1, 0, 0)$$

$$3 \rightarrow (0, -1, 0, 0, 0)$$

$$4 \rightarrow (1, 0, 0, 0, 0)$$

## Partial characterization

**Proposition.** Let  $W \subseteq C^\omega$  be prefix independent ( $W = CW$ ), and suppose that all games on finite arenas with the winning condition  $(W, \overline{W})$  are positionally determined.

Assume further that  $W$  satisfies the **permutation property**

$$(v\mathbf{x}y w)^\omega \in W \quad \text{iff} \quad (v\mathbf{y}x w)^\omega \in W.$$

Then  $W$  coincides with some **LE** condition on all ultimately periodic words: consequently, the respective games are equivalent.

It is **open** if the permutation property is necessary.

## Further questions

Can we have a similar characterization of **finite-memory** determinacy over finite arenas?

Is there an efficient reduction of mean-payoff games to parity games?

Can we improve upon the complexity of solving mean-payoff games, e.g., to  $n^{\mathcal{O}(\log n)}$  ?

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